Distance and Related Notions in Graphs

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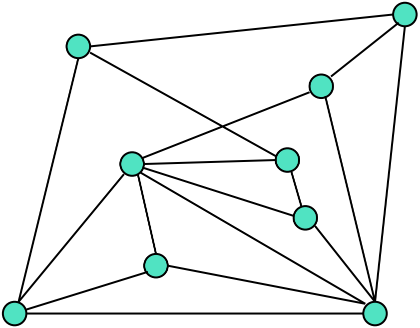
Outline

1. Introduction
2. Definitions
3. Representing Complex Graphs
4. Application to Music
5. References

Background from last talk

Definition (Connected graphs)

An undirected graph $\mathcal{G}$ for which there exists a path between every pair of vertices is said to be *connected*. Each maximal connected piece of a graph is called a *connected component*.

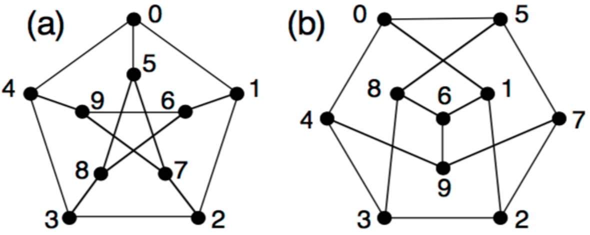


Graph isomorphisms

Definition (Isomorphism)

An *isomorphism of graphs* $\mathcal{G}_1$ and $\mathcal{G}_2$ is an edge-preserving bijection between the vertex sets of $\mathcal{G}_1$ and $\mathcal{G}_2$ such that any two vertices u and v of $\mathcal{G}_1$ are adjacent in $\mathcal{G}_1$ iff $f(u)$ and $f(v)$ are adjacent in $\mathcal{G}_2$.

Two graphs are said to be *isomorphic* if this bijection exists—loosely speaking, if you can draw $\mathcal{G}_1$ to look like $\mathcal{G}_2$.



Metrics

Definition (Metrics)

A *metric* $\mathbb{M}$ on a set X is a (distance) function $\mathbb{M} : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$, the following three axioms hold:

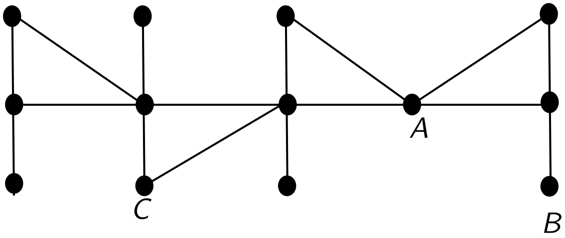
- $\mathbb{M}(x, y) = 0 \iff x = y$ (identity of indiscernibles)
- $\mathbb{M}(x, y) = \mathbb{M}(y, x)$ (symmetry)
- $\mathbb{M}(x, z) \leq \mathbb{M}(x, y) + \mathbb{M}(y, z)$ (triangle inequality)

- **Examples**: Euclidean: $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, Manhattan: $\sum_{i=1}^{n} |x_i - y_i|$, Minkowski: $\sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}$ (with $p \geq 1$), etc.

Distance

Definition (Distance)

For a connected graph $\mathcal{G}$, the *distance* $d(u, v)$ from vertex u to vertex v is the length—ie number of edges—of a shortest u-v path in $\mathcal{G}$ (regardless of the number of alternative paths).



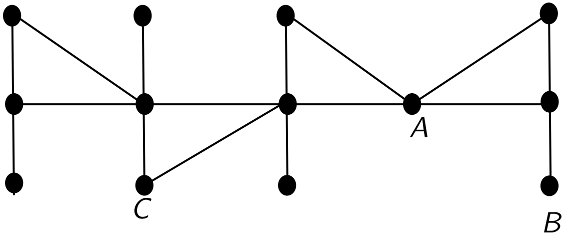
$$d(A, C) = d(A, B) = 2$$

Eccentricity

Definition (Eccentricity)

The greatest distance from a given vertex v to any other vertex x on a given graph is called the graph's *eccentricity*:

$$\mathrm{ecc}(v) = \max_{x \in V(\mathcal{G})} \{d(v, x)\}.$$



$\mathrm{ecc}(A) = \mathrm{ecc}(C) = 3 \qquad \mathrm{ecc}(B) = 5$

Radius, diameter, periphery, centre

For a given graph $\mathcal{G}$,

Definition (Radius, Diameter)

The radius $\mathrm{rad}(\mathcal{G})$ is the value of smallest eccentricity. The diameter $\mathrm{diam}(\mathcal{G})$ is defined as the value of greatest eccentricity.

Definition (Periphery, Centre)

The periphery is the set of vertices V such that $\mathrm{ecc}(V) = \mathrm{diam}(\mathcal{G})$. The centre is the set of vertices V such that $\mathrm{ecc}(V) = \mathrm{rad}(\mathcal{G})$.

Question: For what kind of graphs, if any, are $\mathrm{diam}(\mathcal{G})$ and $\mathrm{rad}(\mathcal{G})$ equal?

Two Properties

Theorem

Every connected graph $\mathcal{G}$ has $\mathrm{rad}(\mathcal{G}) \leq \mathrm{diam}(\mathcal{G}) \leq 2\mathrm{rad}(\mathcal{G})$.

Proof Outline.

We only need show the second inequality, $\mathrm{diam}(\mathcal{G}) \leq 2\mathrm{rad}(\mathcal{G})$. Choose vertices u and v such that $d(u, v) = \mathrm{diam}(\mathcal{G})$, and choose a vertex c in the centre. Then use the triangle inequality. $\square$

Theorem

Every graph of $n \geq 1$ vertices is isomorphic to the centre of some graph.

Motivation

Until now, we have employed mostly hand-wavy methods to understanding graphs and answering natural questions like

- Are two vertices connected by a sequence of edges (if not a single edge)?
- What is the minimum number of edges we need to traverse from one vertex to another?
- What is the minimum number of edges we need to traverse from any vertex on the graph to any other?
- How can we tell if a graph is connected?
- ...

These questions are not hard to answer for smaller graphs, but become more difficult as the number of edges and vertices grows (even with algorithms like Bellman-Ford).

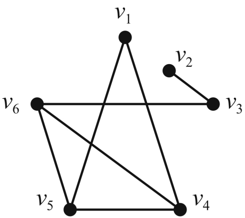
Adjacency matrix

Definition (Adjacency matrix)

Let $\mathcal{G}$ be a graph with vertices $v_1, v_2, \ldots, v_n$. Then the *adjacency matrix* of $\mathcal{G}$ is the $n \times n$ matrix A whose (i, j) entry, denoted by $[A]_{i,j}$, is defined by

$$[A]_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Example:



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

A Few Properties

For adjacency matrices A, A_1, A_2 :

- For undirected graphs, A is symmetric about the major diagonal.
- On simple graphs, the main diagonal of A has all 0's.
- Given two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with respective A_1 and A_2, if there is a permutation of the rows and columns of A_1 that gives A_2, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.
- The sum of the vertices in a row or column equals the degree of the vertex it represents.

Walks and Adjacency Matrices

Theorem

For $\mathcal{G}$ a graph with vertices labelled as $v_1, v_2, \ldots, v_n$, A its corresponding adjacency matrix, and k a positive integer, the (i,j) entry of A^k is equal to the number of walks from v_i to v_j that use exactly k edges.

Proof Outline.

By induction. When $k = 1$, $[A_{i,j}] = 1$—only one edge is available. Let a_{ij} be the ij^{th} entry of A.

Suppose that the ij^{th} entry of A^k, b_{ij}, is the number of k-edge walks from v_i to v_j.The ij^{th} entry of A^{k+1} is then $\sum_{m=1}^{n} a_{im} b_{mj}$. Pick $a_{i1} b_{1j}$; this is the number of k-edge walk from v_1 to v_j times the number of 1-edge walks from v_i to v_1. This is in turn the number of $(k+1)$-edge walks from v_i to v_j. Now choose any other m: the argument still holds. $\square$

Matrix Sum

Definition (Matrix Sum)

Given a graph $\mathcal{G}$ of order n with adjacency matrix A, and given a positive integer k, define the matrix sum S_k to be

$$S_k = \mathbb{I} + A + A^2 + \ldots + A^k,$$

where $\mathbb{I}$ is the $n \times n$ identity matrix.

Motivation.

See previous theorem. □

Eccentricity, Radius, and Diameter

Theorem

Let $\mathcal{G}$ be a connected graph with vertices labelled $v_1, v_2, \ldots, v_n$, and let A be its corresponding adjacency matrix.

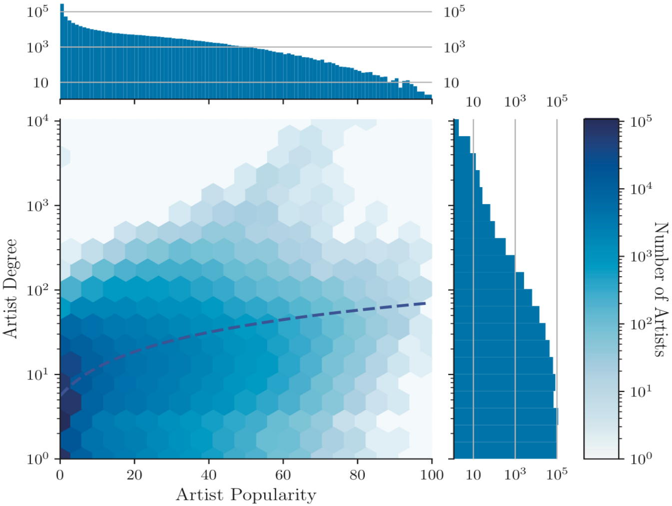
- *If k is the smallest positive integer such that row j of S_k contains no zeros, then $\mathrm{ecc}(v_j) = k$.*
- *If r is the smallest positive integer such that all entries of at least one row of S_r are positive, then $\mathrm{rad}(\mathcal{G}) = r$.*
- *If m is the smallest positive integer such that all entries of S_m are positive, then $\mathrm{diam}(\mathcal{G}) = m$.*

A network analysis of Spotify (by T. South et al)

- A network of all the artists on Spotify connected by who they worked with
- 1,250,065 artists (vertices on undirected graph)
- 3,766,631 collaborations (edges on undirected graph)
- Snowball sampling starting with Kanye West
- Get metadata on these artists (eg popularity, etc)
- Represent using adjacency matrix:

$$A = \begin{array}{c c} & \begin{array}{ccc} \text{Kanye} & \text{Drake} & \text{Taylor} \end{array} \\ \begin{array}{c} \text{Kanye} \\ \text{Drake} \\ \text{Taylor} \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Relative popularity



Eigenvector centrality

- Calculate the *eigenvector centrality*: This is just taking the eigenvector with the largest corresponding eigenvalue

$$Av = \lambda v$$

A

Filter popularity + eigenvector centrality

- Take the most popular contemporary artists

B

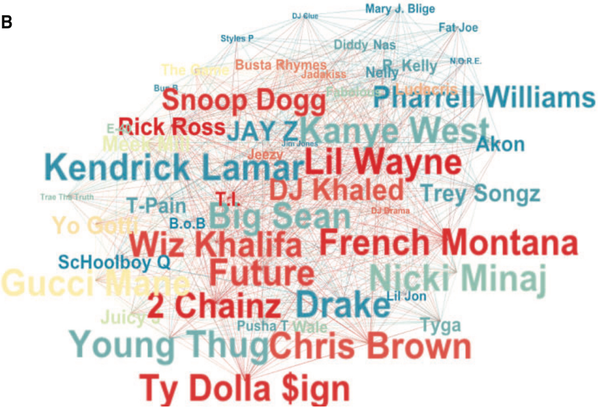
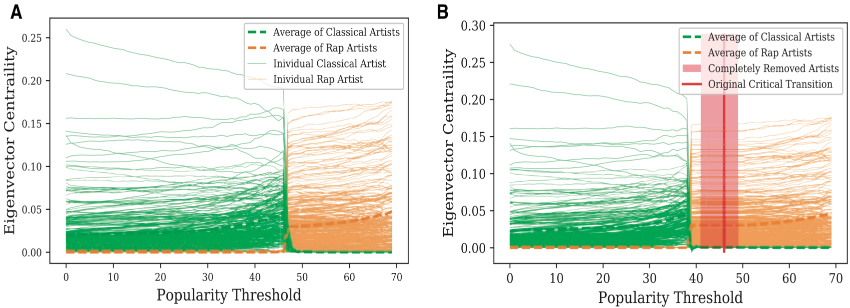


Figure: Dominance shifts from classical music to rappers.

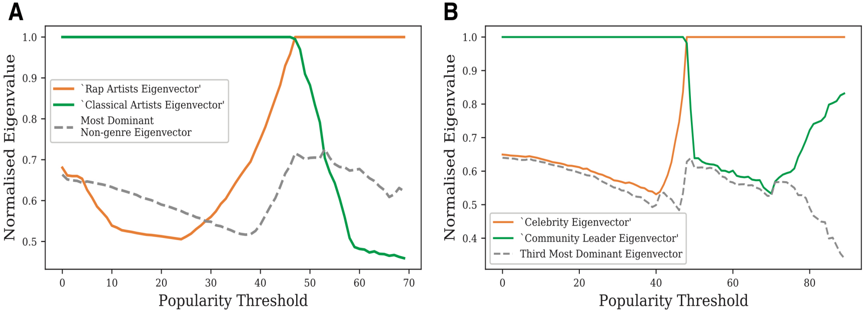
When does it all change?

- Take the network and chop off nodes with a popularity of 10 or less
- Nothing changes in that critical region, but the location of the transition is shifted
- Which means that the entire structure of the graph changes suddenly as we account for popularity!



Why? It's all in the eigenvalues

- The vectors representing classical dominance and rap dominance exist the entire time, but they change ranking:

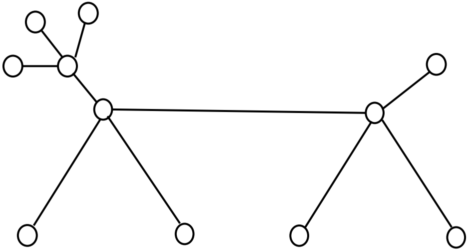


References and further reading

1. BLeversha, Gerry. "Combinatorics and graph theory, by John M. Harris, Jeffry L. Hirst, Michael J. Mossinghoff. Pp. 225.£ 24 (hb). 2000. ISBN 0 387 98736 3 (Springer-Verlag)." *The Mathematical Gazette* 86.505 (2002): 177-178.

2. South, Tobin, Matthew Roughan, and Lewis Mitchell. "Popularity and centrality in Spotify networks: critical transitions in eigenvector centrality." *Journal of Complex Networks* 8.6 (2020): cnaa050.

3. Bryan, Kurt, and Tanya Leise. "The $25,000,000,000 eigenvector: The linear algebra behind Google." *SIAM review* 48.3 (2006): 569-581.



This graph is well known for its bark.